Research Program in Game Theory and Mathematical Economics

Research Memorandum No. 28

June 1967

ON BALANCED GAMES WITH INFINITELY MANY PLAYERS

by

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Let S be an arbitrary set, and Σ a field of
subsets of S.[*]   Let v be a bounded real function
defined on Σ with non-negative values, such that
$v(\emptyset) = 0$ and $v(S) > 0.$ We shall call the triple
$[S,\Sigma,v]$ a game; S is the set of players, Σ the set of
coalitions and v is the payoff[**] function.   An outcome
of the game is a bounded additive real function λ defined
on Σ, for which $\lambda(S) = v(S).$ If an outcome λ fulfils
$\lambda(A) \geq v(A)$ for each $A \in \Sigma,$ it belongs, by definition,
to the core of the game.   For $A \in \Sigma,$ let χ_A be the
characteristic function of A, i.e. $\chi_A(s) = 1$ if $s \in A$
and $\chi_A(s) = 0$ if $s \in S \setminus A,$ for all $s \in S.$ A game
is balanced if

$$\sup\Sigma_i a_i v(A_i) \leq v(S) ,$$

when the sup is taken over all finite sequences of a_i and
A_i, where the a_i are non-negative numbers, the A_i are
in Σ, and $\Sigma_i a_i \chi_{A_i} \leq \chi_S.$[***]

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[*] Σ fulfils: (a) $\emptyset \in \Sigma.$ (b) $A \in \Sigma \Rightarrow S \setminus A \in \Sigma.$
(c) $A \in \Sigma, B \in \Sigma \Rightarrow A \cup B \in \Sigma.$

[**]   v is often called the "characteristic function" of
the game.   We refrain from that terminology because
the same term is used with a different meaning in this
paper.

[***]   It is easy to verify that this sup does not change
even if it is constrained by $\Sigma_i a_i \chi_{A_i} = \chi_S$ (instead of
the inequality); also, for balanced games, the sup
equals $v(S).$

This concept is due to L.S. Shapley [2], who proved that a finite game (S is finite) has a non-empty core if and only if it is balanced. In this paper we extend this result to an arbitrary set S.

U. Liberman [3] dealt with the case when (S, Σ, μ) is a finite, separable, non-atomic measure space (μ is a measure). He required, in addition to the balancedness condition, that the payoff function v be continuous on (Σ, ρ) where ρ is the metric induced on Σ by μ. Then, using Shapley's theorem for the finite case, he proved the existence of a measure in the core, absolutely continuous w.r.t. μ. We prove such a result with weaker assumptions.

The author wishes to thank Professor R.J. Aumann and Dr. B. Peleg for some helpful conversations.

<u>The Main Theorem</u> A game has a non-empty core iff it is balanced.

<u>Proof</u> We shall show that a balanced game has a non-empty core. The other side of the implication is easily verified.

Let **X** denote the linear space of all finite real combinations of characteristic functions of sets in Σ. (The completion of X in the sup metric is denoted in Dunford and Schwartz [1] by $B(S, \Sigma)$.) For non-negative vectors x in **X**, (i.e. $x(s) \geq 0$ for $s \in S$) we define:

$$p(x) = \sup \Sigma_i a_i v(A_i) ,$$

where the sup is taken over all finite sequences of a_i and A_i, where the a_i are non-negative real numbers, the A_i are in Σ, and $\Sigma_i a_i \chi_{A_i} \leq \chi_S$.

Let X^+ denote the positive cone of X, i.e.

$$X^+ = \{x \in X \mid x \geq 0\} .$$

Then p is a super-additive positive-homogeneous [****] functional on X^+. We shall prove the existence of a linear functional F on X for which $F(x) \geq p(x)$ when $x \in X^+$, and $F(\chi_S) = v(S)$. Naturally, the set function induced by F on Σ is in the core of the game. The idea behind the proof is closely related to the Hahn-Banach theorem. (See, for instance, [1]).

Let Y be a subspace of X containing χ_S, and let $Y^+ = Y \cap X^+$. Assume that a linear functional F is defined on Y for which $F(x) \geq p(x)$ when $x \in Y^+$, and $F(\chi_S) = v(S)$. If $Y \neq X$, there is a set A in Σ such that $\chi_A \notin Y$. Define $Z = \text{span}\{Y \cup \{\chi_A\}\}$. Every vector in Z has a unique representation in the form $y + a\chi_A$ with $y \in Y$. For any real c the function G defined on Z by the equation $G(y + a\chi_A) = F(y) + ac$ is a proper extension of F. It remains to show that c may be chosen so that $G(y + a\chi_A) \geq p(y + a\chi_A)$ when $y + a\chi_A \geq 0$. Because of the positive-homogeneity of p, it is sufficient to prove the

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[****] A functional p is called super-additive if $p(x+y) \geq p(x) + p(y)$. It is called positive-homogeneous if $ap(x) = p(ax)$ when $a \geq 0$.

existence of c, such that for any y,z in Y, $y + \chi_A \geq 0$ and $z - \chi_A \geq 0$ imply $F(y) + c \geq p(y + \chi_A)$ and $F(z) - c \geq p(z - \chi_A)$. The last two inequalities are equivalent to

$$F(z) - p(z - \chi_A) \geq c \geq p(y + \chi_A) - F(y).$$

So it is sufficient to prove:

$$F(z) - p(z - \chi_A) \geq p(y + \chi_A) - F(y)$$

or,

$$F(z) + F(y) \geq p(z - \chi_A) + p(y + \chi_A).$$

But $y + \chi_A \geq 0$ and $z - \chi_A \geq 0$ imply $y + z \in Y^+$, so

$$p(z - \chi_A) + p(y + \chi_A) \leq p(z + y) \leq F(z + y) = F(z) + F(y)$$

and the desired inequality is proved. The proof is completed by a standard use of Zorn's lemma. Q.E.D.

Next we deal with the problem of existence of a σ-additive outcome in the core, assuming that Σ is a σ-field.

A necessary and sufficient condition for an additive set function λ to be σ-additive is that $\lambda(A_i) \to \lambda(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ with $\cup_{i=1}^{\infty} A_i = S$. If for every such sequence $v(A_i) \to v(S)$ and λ is in the core, i.e. $\lambda(A) \geq v(A)$ $A \in \Sigma$, we can easily conclude the desired condition $\lambda(A_i) \to v(S) = \lambda(S)$. So we have proved:

Lemma A If $v(A_i) \to v(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ, the union of which is S, then every outcome in the core is σ-additive.

Indeed we know a little bit more. If λ belongs to the core, then $\lambda(A) \geq p(A)$, $A \in \Sigma$ and we get a somewhat stronger result:

Lemma B If $p(A_i) \to p(S)$ for any monotone increasing sequence $\{A_i\}_{i=1}^{\infty}$ in Σ, the union of which is S, then every outcome in the core is σ-additive.

Of course the second condition is not necessary for the existence of a σ-additive outcome in the core. For example let:

$$v(A) = \begin{cases} 1 & A = S \\ 0 & \text{otherwise} \end{cases}$$

So two open questions may be asked: Is the condition of lemma B necessary that every outcome in the core should be σ-additive? and what is a necessary and sufficient condition for the existence of a σ-additive outcome in the core? (Assuming the core is non-empty).

The treatment of another problem was found to be more successful. Assume a game $[S, \Sigma, v]$ and an additive function μ on Σ. What is the "continuity" condition on v with respect to μ, such that every outcome in the core will be "continuous" with respect to μ?

If $v(S \setminus A) = v(S)$ then for each λ in the core $\lambda(A) = 0$; otherwise

$$v(S) = v(S \setminus A) \leqslant \lambda(S \setminus A) = \lambda(S) - \lambda(A) < \lambda(S) = v(S) ,$$

a contradiction. On the other hand, if λ is in the core and $\lambda(A) = 0$, then $v(A) = 0$. We can state this result as follows:

<u>Lemma C</u> (i) If $v(S \setminus A) = v(S)$ for every μ-null set A, then any outcome in the core vanishes on the μ-null sets.

(ii) If an outcome in the core vanishes on μ-null sets, then v does the same.

Another similar simple result is given below:

<u>Lemma D</u> If v fulfils the conditions of lemma B and lemma C (i), then any outcome in the core is absolutely continuous w.r.t. μ.

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
THE HEBREW UNIVERSITY OF JERUSALEM	Unclassified
	2b. GROUP

3. REPORT TITLE

ON BALANCED GAMES WITH INFINITELY MANY PLAYERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (Last name, first name, initial)

SCHMEIDLER, DAVID

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1967	7	3

8a. CONTRACT OR GRANT NO. F61052 67 C 0094	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. RR 003 0701	Research Memorandum No. 28
c. Task No. NR 047-045	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Logistics & Mathematical Statistics Branch, Office of Naval Research, Washington, D.C. 20360

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A cooperative game with side payments with an arbitrary set of players is formulated. The notions of balancedness and core remain unchanged in this general game. L.S. Shapley invented the two notions above, and proved that the core is non-empty if and only if the game is balanced. In this memorandum the same result is proved for a game with an arbitrary set of players.

DD FORM 1473 (1 JAN 64)

14. KEY WORDS	LINK A		LINK B		LINK C	
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